COMPACTNESS OF THE SOLUTION OPERATOR TO $\overline{\partial}$ IN WEIGHTED L^2 - SPACES.

FRIEDRICH HASLINGER AND BERNARD HELFFER

ABSTRACT.
In this paper we discuss compactness of the canonical solution operator to $\overline{\partial}$ on weigthed L^2 spaces on $\mathbb{C}^n$. For this purpose we apply ideas which were used for the Witten Laplacian in the real case and various methods of spectral theory of these operators. We also point out connections to the theory of Dirac and Pauli operators.

1. INTRODUCTION.

1.1. **Background for bounded pseudoconvex domains.**

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. We consider the $\overline{\partial}$-complex

$$L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\overline{\partial}} 0 \,,$$

where $L^2_{(0,q)}(\Omega)$ denotes the space of $(0,q)$-forms on Ω with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial}\left(\sum_J {}' a_J \, d\overline{z}_J\right) \sum_{j=1}^n \sum_J {}' \frac{\partial a_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J,$$

where $\sum {}'$ means that the sum is taken only over increasing multi-indices J.
The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right hand side belongs to $L^2_{(0,q+1)}(\Omega)$. Then $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ denoted by $\overline{\partial}^*$.
The complex Laplacian $\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}$ acts as an unbounded selfadjoint operator on $L^2_{(0,q)}(\Omega)$, $1 \leq q \leq n$, it is surjective and therefore has a continuous inverse, the $\overline{\partial}$-Neumann operator N_q. If v is a closed $(0, q + 1)$-form, then $\overline{\partial}^* N_{q+1}v$ provides the canonical solution to $\overline{\partial}u = v$, which is orthogonal to the kernel of $\overline{\partial}$ and so has minimal norm (see for instance [ChSh]).

A survey of the L^2-Sobolev theory of the $\overline{\partial}$-Neumann problem is given in [BS].

The question of compactness of N_q is of interest for various reasons. For example, compactness of N_q implies global regularity in the sense of preservation of Sobolev spaces [KN]. Also, the Fredholm theory of Toeplitz operators is an immediate consequence of compactness in the $\overline{\partial}$-Neumann problem [V], [HI], [CD]. There are additional ramifications for certain C^*-algebras naturally associated to a domain in $\mathbb{C}^n$ [SSU]. Finally, compactness is a more robust property than global regularity - for example, it localizes, whereas global regularity does not - and it is generally believed to be more tractable than global regularity.

A thorough discussion of compactness in the $\overline{\partial}$-Neumann problem can be found in [FS2]. The study of the $\overline{\partial}$-Neumann problem is essentially equivalent to the study of the canonical solution operator to $\overline{\partial}$:

The $\overline{\partial}$-Neumann operator N_q is compact from $L^2_{(0,q)}(\Omega)$ to itself if and only if the canonical solution operators

$$\overline{\partial}^* N_q : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q-1)}(\Omega) \ \ \text{and} \ \ \overline{\partial}^* N_{q+1} : L^2_{(0,q+1)}(\Omega) \longrightarrow L^2_{(0,q)}(\Omega)$$

are compact.

Interestingly, in many situations, the restriction of the canonical solution operator to forms with *holomorphic* coefficients arises naturally [SSU], [FS1]. Compactness of the restriction to forms with holomorphic coefficients already implies compactness of the original solution operator to $\overline{\partial}$ in the case of convex domains, see [FS2]. There are many examples for non-compactness, where the obstruction already occurs for forms with holomorphic coefficients (see [Has1], [Has2], [Kr] and [L]).

In [CD] it is shown that compactness of the $\overline{\partial}$-Neumann operator implies compactness of the commutator $[P, M]$, where P is the Bergman projection and M is pseudodifferential operator of order 0. In [Has4] it is shown that compactness of the canonical solution operator to $\overline{\partial}$ restricted to $(0, 1)$-forms with holomorphic coefficients implies compactness of the commutator $[P, M]$ defined on the whole $L^2(\Omega)$.

Let $A^2_{(0,1)}(\Omega)$ denote the space of all $(0, 1)$-forms with holomorphic coefficients belonging to $L^2(\Omega)$.

Another result from [Has4] states that for a bounded pseudoconvex domain Ω the $\overline{\partial}$-Neumann operator

$$N_1 : L^2_{(0,1)}(\Omega) \longrightarrow L^2_{(0,1)}(\Omega)$$

restricted to $(0, 1)$-forms with holomorphic coefficients can be written in the form

$$\mathcal{P}N\mathcal{P}f = \sum_{k=1}^{n} [P, M_k] \left(\sum_{j=1}^{n} [\overline{M}_j, P] f_j \right) d\overline{z}_k$$

here $\mathcal{P} : L^2_{(0,1)}(\Omega) \longrightarrow A^2_{(0,1)}(\Omega)$ denotes the componentwise projection and M_j and $\overline{M}_j$ denotes the multiplication by z_j and $\overline{z}_j$ respectively.

The restriction of the canonical solution operator to forms with holomorphic coefficients has many interesting aspects, which in most cases correspond to certain growth properties of the Bergman kernel.

In [Has1] the canonical solution operator S_1 to $\overline{\partial}$ restricted to $(0, 1)$-forms with holomorphic coefficients is investigated.

It is shown that the canonical solution operator $S_1 : A^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega)$ has the form

$$S_1(g)(z) = \int_\Omega B(z,w) < g(w), z - w > d\lambda(w),$$

where B denotes the Bergman kernel of Ω and

$$< g(w), z - w >= \sum_{j=1}^n g_j(w)(\overline{z}_j - \overline{w}_j),$$

for $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$; it can also be written in the form

$$S_1 g = \sum_{j=1}^n [\overline{M}_j, P] g_j.$$

It follows that the canonical solution operator is a Hilbert Schmidt operator for the unit disc $\mathbb{D}$ in $\mathbb{C}$, but fails to be Hilbert Schmidt for the unit ball in $\mathbb{C}^n$, $n \geq 2$ (see also [LY]).

1.2. **The case of unbounded domains.**

Not very much is known in the case of unbounded domains. In this paper we discuss the compactness of the canonical solution operator to $\overline{\partial}$ on weighted L^2-spaces over $\mathbb{C}^n$. We define

$$L^2(\mathbb{C}^n, \varphi) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \ : \int_{\mathbb{C}^n} |f(z)|^2 \exp(-2\varphi(z)) \, d\lambda(z) < \infty\},$$

where φ is a suitable weight-function.
There is an interesting connection of $\overline{\partial}$ with the theory of Schrödinger operators with magnetic fields, see for example [Ch], [B], [FS3] and [ChF] for recent contributions exploiting this point of view.
For the case of one complex variable results of Helffer-Mohamed [HeMo], Iwatsuka [I] and Shen [She1] can be used to discuss compactness of the canonical solution operator to $\overline{\partial}$. For instance, if $\varphi(z) = |z|^2$, then the canonical solution operator $S : L^2(\mathbb{C}, \varphi) \longrightarrow L^2(\mathbb{C}, \varphi)$ to $\overline{\partial}$ fails to be compact. If $\Delta\varphi(z) \to \infty$ as $|z| \to \infty$, then the canonical solution operator $S : L^2(\mathbb{C}, \varphi) \longrightarrow L^2(\mathbb{C}, \varphi)$ to $\overline{\partial}$ is compact ([Has3]).
In this paper we first give a necessary and sufficient condition in terms of the weight function φ in the complex one-dimensional case for the solution operator to be compact on $L^2(\mathbb{C}, \varphi)$ continuing the work from [Has3] and using results from [AuBe], [HeMo], [I], [She1], and [St].
In the case of several complex variables, we meet an obvious condition for solving $\overline{\partial}u = f$. The $(0, 1)$-form f should satisfy $\overline{\partial}f = 0$. So we are asking for the existence of a continuous operator S^{can}, which will be called the canonical solution operator :

$$(1.1) \qquad L^2_{(0,1)}(\mathbb{C}^n, \varphi) \cap \text{ Ker } \overline{\partial} \ni f \mapsto u = S^{can} f \in L^2(\mathbb{C}^n, \varphi) \cap (\text{ Ker } \overline{\partial})^\perp,$$

giving the minimal solution of the problem.
When the weight function φ is plurisubharmonic, we will for example show that the condition that the lowest eigenvalue λ_φ of the Levi matrix M_φ satisfies

$$\lim_{|z| \to \infty} \lambda_\varphi(z) = +\infty$$

implies the existence of the canonical solution operator and its compactness.

For decoupled weights

$$\varphi(z) = \varphi_1(z_1) + \varphi_2(z_2) + \cdots + \varphi_n(z_n) \,,$$

the canonical solution operator to $\overline{\partial}$ fails, under very weak additional assumptions to be compact and we will show that it is even true on $A^2_{(0,1)}(\mathbb{C}^n, \varphi)$ (see [Sch]).

There are other interesting connections between $\overline{\partial}$ and Schrödinger operators, see for example the discussion in [B] and between compactness in the $\overline{\partial}$-Neumann problem and property (P) on the one hand, and the asymptotic behavior, in a semi-classical limit, of the lowest eigenvalues of certain magnetic Schrödinger operators and of their non-magnetic counterparts, respectively, on the other ([FS3]). The main result in [FS3] shows that (for certain Hartogs domains in $\mathbb{C}^2$) compactness properties of the $\overline{\partial}$-Neumann operator may be interpreted as a consequence of well known diamagnetic inequalities (originally due to Kato) in the theory of Schrödinger operators (see [LL], [CFKS] and [He1]).

Finally, we also point out some interesting connections to the theory of Dirac and Pauli operators, when discussing the case of non-compact resolvents (see [CFKS], [Er], [HNW], [Roz], [Tha]).

Acknowledgements

2. THE COMPLEX ONE-DIMENSIONAL CASE

Let φ be a subharmonic $\mathcal{C}^2$-function. We want to solve $\overline{\partial} u = f$ for $f \in L^2(\mathbb{C}, \varphi)$. The canonical solution operator to $\overline{\partial}$ gives a solution with minimal $L^2(\mathbb{C}, \varphi)$-norm. We substitute $v = u \, e^{-\varphi}$ and $g = f \, e^{-\varphi}$ and the equation becomes

$$\overline{D}v = g \,,$$

where

$$(2.1) \qquad\qquad \overline{D} = e^{-\varphi} \frac{\partial}{\partial \overline{z}} \, e^{\varphi}.$$

u is the minimal solution to the $\overline{\partial}$-equation in $L^2(\mathbb{C}, \varphi)$ if and only if v is the solution to $\overline{D}v = g$ which is minimal in $L^2(\mathbb{C})$.
The formal adjoint of $\overline{D}$ is

$$(2.2) \qquad\qquad D = -e^{\varphi} \frac{\partial}{\partial z} e^{-\varphi}.$$

Let us introduce

$$(2.3) \qquad\qquad \mathcal{S} = \overline{D}D \,.$$

Since $\overline{D} = \frac{\partial}{\partial \overline{z}} + \frac{\partial \varphi}{\partial \overline{z}}$ and $D = -\frac{\partial}{\partial z} + \frac{\partial \varphi}{\partial z}$, we see that

$$\mathcal{S} = -\frac{\partial^2}{\partial z \partial \overline{z}} - \frac{\partial \varphi}{\partial \overline{z}} \frac{\partial}{\partial z} + \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial \overline{z}} + \left| \frac{\partial \varphi}{\partial z} \right|^2 + \frac{\partial^2 \varphi}{\partial z \partial \overline{z}} \,.$$

So

$$(2.4) \qquad \mathcal{S} = \frac{1}{4}\left(-\Delta_A + B\right),$$

where the 1-form $A = A_1\,dx + A_2\,dy$ is related to the weight φ by

$$(2.5) \qquad A_1 = -\partial_y\varphi\,,\ A_2 = \partial_x\varphi\,,$$

$$(2.6) \qquad \Delta_A = \left(\frac{\partial}{\partial x} - iA_2\right)^2 + \left(\frac{\partial}{\partial y} + iA_1\right)^2\,,$$

and the magnetic field $Bdx \wedge dy$ satisfies

$$(2.7) \qquad B(x,y) = \Delta\varphi(x,y)\,.$$

Hence $\mathcal{S}$ is (up to a multiplicative constant) a Schrödinger operator with magnetic field and an electric potential B. In addition, we know from [Sima] that this operator is essentially self-adjoint on $C_0^\infty(\mathbb{C})$.

In [Has3] (completing a result of M. Christ [Ch]), a link was established between the compactness of the canonical solution operator to $\bar{\partial}$ and the properties of the resolvent of $\mathcal{S}$. In this setting it was supposed that the weight functions φ are in the class $\mathcal{W}$.

Definition 2.1.
We say that φ is in the class $\mathcal{W}$ if :
   (1) *$\nu = \Delta\varphi dx$, is a doubling measure, which means that there exists a constant C such that for all $z \in \mathbb{C}$ and $r \in \mathbb{R}^+$,*

$$\nu(B(z,2r)) \le C\,\nu(B(z,r)),$$

   where $B(z,r)$ denotes the ball with center z and radius r ;
   (2) *there exists a constant $\delta > 0$ such that for all $z \in \mathbb{C}$,*

$$\nu(B(z,1)) \ge \delta.$$

In fact Marco, Massaneda and Ortega-Cerda [MMO] (Theorem C, p. 884) found out that already condition (1) in the last definition implies that the canonical solution operator to $\bar{\partial}$ is continuous. Hence it follows from [Has3]

Theorem 2.2.
Let φ be a subharmonic $\mathcal{C}^2$-function on $\mathbb{R}^2$ such that $\Delta\varphi$ defines a doubling measure. The canonical solution operator $S : L^2(\mathbb{C},\varphi) \longrightarrow L^2(\mathbb{C},\varphi)$ to $\bar{\partial}$ is compact if and only if $\mathcal{S}$ has compact resolvent.

Now we prove a criterion of compactness, which can be expressed in terms of the weight function φ only. Here we extend a result due to Helffer and Morame ([HeMor]) based on methods developed by Iwatsuka ([I]) and Shen ([She1]). For this purpose we assume the stronger condition that the weight function φ is a subharmonic $\mathcal{C}^2$ function and that $\Delta\varphi$ belongs to the reverse Hölder class $B_2(\mathbb{R}^2)$ consisting of L^2 positive and almost non zero everywhere functions V for which there exists a constant $C > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q V^2\,dx\right)^{\frac{1}{2}} \le C\left(\frac{1}{|Q|}\int_Q V\,dx\right)$$

for any ball Q in $\mathbb{R}^2$.

It is known that if V is in B_q for some $q > 1$ then V is in the Muckenhoupt class A_∞ and the corresponding measure $V(x)dx$ is doubling. More precisely it is known from [St] that

$$A_\infty = \cup_{q>1} B_q \ .$$

Note that any positive (non zero) polynomial is in B_q for any $q > 1$.

Theorem 2.3.
Let φ be a subharmonic $\mathcal{C}^2$- function on $\mathbb{R}^2$ such that

$$(2.8) \qquad\qquad\qquad \Delta\varphi \in B_2(\mathbb{R}^2) \ .$$

Then the canonical solution operator $S : L^2(\mathbb{C}, \varphi) \longrightarrow L^2(\mathbb{C}, \varphi)$ to $\overline{\partial}$ is compact if and only if

$$(2.9) \qquad\qquad \lim_{|z|\to\infty} \int_{B(z,1)} \Delta\varphi(y)\, dy = +\infty \ .$$

Proof.
Using Theorem 2.2, we have just to analyze if $-\Delta_A + \Delta\varphi$ has compact resolvent. Using the standard comparison between selfadjoint operators :

$$(2.10) \qquad\qquad -2\Delta_A \geq -\Delta_A + \Delta\varphi \geq -\Delta_A$$

we observe that $-\Delta_A + \Delta\varphi$ has compact resolvent if and only if $-\Delta_A$ has compact resolvent.

In one direction, we can apply a result of Iwatsuka ([I] , Theorem 5.2) which says

Proposition 2.4.
Suppose that $A \in H^1_{loc}$ and that $-\Delta_A$ has compact resolvent. Then

$$(2.11) \qquad\qquad \lim_{|z|\to\infty} \int_{B(z,1)} B(y)^2\, dy = +\infty \ ,$$

with $B = curlA$.

Iwatsuka adds a C^∞ assumption on the magnetic potential. But at least in the two dimensional case, one can use properties of the Curl operator as mentioned in Appendix I of [T], in order to release this assumption. Note that in our case $B = \Delta\varphi$. By the definition of the reverse Hölder class $B_2(\mathbb{R}^2)$, (2.11) implies (2.9).

For the other direction, we first use a version of the diamagnetic property for Schrödinger operators (see for example [KS] Cor. 1.4) saying that :
If $-\Delta + \Delta\varphi$ has compact resolvent, then $-\Delta_A + \Delta\varphi$ has compact resolvent. So it is enough to prove that $-\Delta + V$ has compact resolvent with $V = \Delta\varphi$.

By the Main Theorem in [I], it suffices to show that

$$(2.12) \qquad\qquad \lim_{|z|\to\infty} \lambda_{0,V}(B(z, 1)) = +\infty \ ,$$

where $\lambda_{0,V}(B(z, 1))$ is the lowest eigenvalue of the Dirichlet realization of $-\Delta + V$ in $B(z, 1)$. Without loss of generality, we can consider, instead of balls, cubes. In this case we use the following improved version of the Fefferman-Phong Lemma as given in [AuBe].

Lemma 2.5.
If $V \in A_\infty$, then there exists $C_V > 0$ and $\beta_V \in]0,1[$ such that, for all cubes Q (with sidelength R), for all $u \in C_0^\infty(Q)$,

$$(2.13) \qquad C_V \frac{m_\beta(R^2 \Theta_Q)}{R^2} \int |u(y)|^2 \, dy \leq \int (|\nabla u(y)|^2 + V(y)|u(y)|^2) \, dy$$

where

$$\Theta_Q = \frac{1}{|Q|} \int_Q V(y) dy \,,$$

and

$$m_\beta(t) = t \text{ for } t \leq 1 \,, \quad \text{and } m_\beta(t) = t^{\beta_V} \text{ for } t \geq 1 \,.$$

We apply Lemma 2.5 with $R = 1$ and $V = \Delta\varphi$. (2.13) gives a lower bound for $\lambda_{0,V}(Q)$ by $C_V \Theta_Q^{\beta_V}$ each time that $\Theta_Q \geq 1$. Therefore Assumption (2.9) implies (2.12) and we are done. $\qquad \Box$

Remark 2.6.
As a variant of the proof, we have the following statement. Suppose that $\Delta\varphi$ belongs to A_∞ (at ∞, i.e. for all the balls meeting the complement of a compact K) and that

$$\liminf_{|z| \to \infty} \int_{B(z,1)} \Delta\varphi(y) dy > 0 \,,$$

then the canonical solution operator S is well defined and Theorem 2.2 is true.
Note that we have also shown that if

$$\lim_{|z| \to \infty} \int_{B(z,1)} \Delta\varphi(y) \, dy = +\infty \,,$$

then S is compact.

We learn from Z. Shen, that, in this 2-dimensional case, one can, by other techniques developed in [She2], improve the necessary part due to Iwatsuka and deduce the same result under the weaker assumption that $\Delta\varphi \in A^\infty$. This proof is much more involved and strongly limited to the two-dimensional case.

3. THE $\bar{\partial}$-EQUATION IN WEIGHTED L^2 - SPACES OF SEVERAL COMPLEX VARIABLES : THE CANONICAL SOLUTION OPERATOR.

Here we apply ideas which were used in the analysis of Witten Laplacian in the real case, see [HeNi].
Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a C^2-weight function and define the space

$$L^2(\mathbb{C}^n, \varphi) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \ : \ \int_{\mathbb{C}^n} |f|^2 \, e^{-2\varphi} \, d\lambda < \infty\},$$

the space $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$ and the space $L^2_{(0,2)}(\mathbb{C}^n, \varphi)$ of $(0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$.
Let $A^2(\mathbb{C}^n, \varphi)$ denote the space of entire functions belonging to $L^2(\mathbb{C}^n, \varphi)$.

We consider the $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n, \varphi) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\mathbb{C}^n, \varphi) \xrightarrow{\overline{\partial}} L^2_{(0,2)}(\mathbb{C}^n, \varphi) .$$

For $v \in L^2(\mathbb{C}^n)$, let

$$\overline{D}_1 v = \sum_{k=1}^n \left(\frac{\partial v}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial \overline{z}_k} v \right) d\overline{z}_k$$

and for $g = \sum_{j=1}^n g_j \, d\overline{z}_j \in L^2_{(0,1)}(\mathbb{C}^n)$, let

$$\overline{D}_1^* g = \sum_{j=1}^n \left(\frac{\partial \varphi}{\partial z_j} g_j - \frac{\partial g_j}{\partial z_j} \right) ,$$

where the derivatives are taken in the sense of distributions.

It is easy to see that $\overline{\partial} u = f$ for $u \in L^2(\mathbb{C}^n, \varphi)$ and $f \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ if and only if $\overline{D}_1 v = g$, where $v = u \, e^{-\varphi}$ and $g = f \, e^{-\varphi}$. It is also clear that the necessary condition $\overline{\partial} f = 0$ for solvability holds if and only if $\overline{D}_2 g = 0$ holds. Here

$$\overline{D}_2 g = \sum_{j,k=1}^n \left(\frac{\partial g_j}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial \overline{z}_k} g_j \right) d\overline{z}_k \wedge d\overline{z}_j.$$

So the existence and the analysis of the canonical solution operator introduced in (1.1) is equivalent to the existence and the analysis of the canonical solution operator for $\overline{D}$, the equivalence being given by

(3.1) $$S_\varphi^{can} = \exp(-\varphi) \, S^{can} \, \exp(\varphi) .$$

We consider the corresponding $\overline{D}$-complex with in particular :

$$L^2(\mathbb{C}^n) \underset{\overline{D}_1^*}{\overset{\overline{D}_1}{\rightleftarrows}} L^2_{(0,1)}(\mathbb{C}^n) \underset{\overline{D}_2^*}{\overset{\overline{D}_2}{\rightleftarrows}} L^2_{(0,2)}(\mathbb{C}^n) .$$

The $\Box$- Laplacians $\Box_\varphi^{(0,0)}$ and $\Box_\varphi^{(0,1)}$ are defined by

(3.2) $$\begin{aligned} \Box_\varphi^{(0,0)} &= \overline{D}_1^* \overline{D}_1 \ , \\ \Box_\varphi^{(0,1)} &= \overline{D}_1 \overline{D}_1^* + \overline{D}_2^* \overline{D}_2 . \end{aligned}$$

It follows that for $g = \sum_{j=1}^n g_j \, d\overline{z}_j$ we have that $\Box_\varphi^{(0,1)} g$ equals

$$\sum_{k=1}^n \left[\sum_{j=1}^n \left(2\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} g_j - \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_j} g_k - \frac{\partial^2 g_k}{\partial z_j \partial \overline{z}_j} + \frac{\partial g_k}{\partial \overline{z}_j} \frac{\partial \varphi}{\partial z_j} - \frac{\partial g_k}{\partial z_j} \frac{\partial \varphi}{\partial \overline{z}_j} + \frac{\partial \varphi}{\partial z_j} \frac{\partial \varphi}{\partial \overline{z}_j} g_k \right) \right] d\overline{z}_k$$

and that

(3.3) $$\Box_\varphi^{(0,1)} = \Box_\varphi^{(0,0)} \otimes I + 2M_\varphi,$$

where

(3.4) $$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{jk} .$$

For φ in $\mathcal{C}^2$, it can be shown (by an extension of a criterion of Simader [Sima]) that $\Box_\varphi^{(0,1)}$ can be extended to a densely defined self-adjoint operator on $L^2_{(0,1)}(\mathbb{C}^n)$, which is again denoted by $\Box_\varphi^{(0,1)}$.

We can now state a natural, rather standard, existence theorem for the canonical operator.

Theorem 3.1.
Let us assume that

$$0 \notin \sigma(\square_\varphi^{(0,1)}) \ .$$

Then, if N_φ denotes its inverse, the operator

$$S_\varphi := (\overline{D}_1)^* N_\varphi \ ,$$

is continuous from $L^2_{(0,1)}(\mathbb{C}^n)$ into $L^2(\mathbb{C}^n)$ and its restriction to $Ker \, D_2$ gives the canonical solution operator S_φ^{can}, hence S^{can} via (3.1).
Proof.
We have :

$$
\begin{aligned}
\langle S_\varphi^* S_\varphi v \, , \, v \rangle &= \langle N_\varphi \overline{D}_1 \overline{D}_1^* N_\varphi v \, , \, v \rangle \\
&= \langle \overline{D}_1 \overline{D}_1^* N_\varphi v \, , \, N_\varphi v \rangle \\
&\leq \langle \overline{D}_1 \overline{D}_1^* N_\varphi v \, , \, N_\varphi v \rangle + \langle \overline{D}_2^* \overline{D}_2 N_\varphi v \, , \, N_\varphi v \rangle \\
&= \langle N_\varphi v \, , \, v \rangle \ .
\end{aligned}
$$

Hence

$$(3.5) \qquad ||S_\varphi v||^2 = \langle S_\varphi^* S_\varphi v \, , \, v \rangle \leq \langle N_\varphi v \, , \, v \rangle \ . $$

$\square$

We also indicate that

$$(3.6) \qquad 4 \, \square_\varphi^{(0,0)} = \Delta_\varphi^{(0)} - \Delta\varphi,$$

where

$$\Delta_\varphi^{(0)} = -\sum_{j=1}^n \left(\left(\frac{\partial}{\partial x_j} + i \frac{\partial\varphi}{\partial y_j} \right)^2 + \left(\frac{\partial}{\partial y_j} - i \frac{\partial\varphi}{\partial x_j} \right)^2 \right)$$

and

$$\Delta\varphi = \sum_{j=1}^n \left(\frac{\partial^2\varphi}{\partial x_j^2} + \frac{\partial^2\varphi}{\partial y_j^2} \right) \ .$$

4. ABOUT GENERAL CRITERIA OF COMPACT RESOLVENT

The analysis of the compactness of the canonical solution operator to $\overline{\partial}$ involves the analysis of the compact resolvent property for Schrödinger operators with compact manifold. We recall in this section a theorem due to Helffer-Mohamed ([HeMo]) on compact resolvents of Schrödinger operators with magnetic fields.
We will analyze the problem for the family of operators :

$$(4.1) \qquad P_A = \sum_{j=1}^n (D_{x_j} - A_j(x))^2 \ .$$

Here $D_{x_j} = -i\frac{\partial}{\partial x_j}$ and the magnetic potential $A(x) = (A_1(x), A_2(x), \cdots, A_n(x))$ is supposed to be C^∞. Under these conditions, the operator is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. We note also that it has the form :

$$P_A = \sum_{j=1}^n X_j^2 \,,$$

with

$$X_j = (D_{x_j} - A_j(x)) \,, \ j = 1, \ldots, n \,.$$

Note that with this choice $X_j^* = X_j$. In particular, the magnetic field is recovered by observing that

$$B_{jk} = \frac{1}{i}[X_j, X_k] = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \,, \quad \text{for } j, k = 1, \ldots, n \,.$$

We introduce for $q \geq 1$ the quantities :

(4.2)
$$m_q(x) = \sum_{j<k} \sum_{|\alpha|=q-1} |\partial_x^\alpha B_{jk}(x)| \,.$$

It is easy to reinterpret this quantity in terms of commutators of the X_j's. Let us also introduce

(4.3)
$$m^r(x) = 1 + \sum_{q=0}^r m_q(x) \,.$$

Then the criterion is

Theorem 4.1. *([HeMo])*
Let us assume that there exists r and a constant C such that

(4.4)
$$m_{r+1}(x) \leq C \, m^r(x) \,, \ \forall x \in \mathbb{R}^n \,,$$

and

(4.5)
$$m^r(x) \to +\infty \,, \quad \text{as } |x| \to +\infty \,.$$

Then P_A has a compact resolvent.

(see also [She1] and [KS] for further results in this direction.)

We will mainly apply this result for the case of real dimension $2n$, where we will write the elements of $\mathbb{R}^{2n}$ in the form $(x_1, y_1, \ldots, x_n, y_n)$ and for the magnetic potential

(4.6)
$$A = \left(-\frac{\partial\varphi}{\partial y_1}, \frac{\partial\varphi}{\partial x_1}, \ldots, -\frac{\partial\varphi}{\partial y_n}, \frac{\partial\varphi}{\partial x_n} \right) \,.$$

5. The analysis of the Laplacian and application

Theorem 5.1.
Let φ be a plurisubharmonic C^2 - function on $\mathbb{C}^n$ such that for the lowest eigenvalue λ_φ of the Levi matrix M_φ the condition

(5.1)
$$\liminf_{|z|\to\infty} \lambda_\varphi(z) > 0 \,,$$

is satisfied. Then the operator $\square_\varphi^{(0,1)}$ has a bounded inverse N_φ on $L^2_{(0,1)}(\mathbb{C}^n)$.

Proof.

For $v = \sum_{k=1}^n v_k \, d\bar{z}_k \in \mathrm{Dom}\,\square_\varphi^{(0,1)}$, we have by (3.3),

$$(5.2) \qquad\qquad \langle \square_\varphi^{(0,1)} v \, , \, v \rangle \geq 2 \langle M_\varphi v \, , \, v \rangle \, .$$

Using Persson's Theorem (see for instance [Ag]), we now conclude from Assumption (5.1) that the bottom of the essential spectrum of $\square_\varphi^{(0,1)}$ is strictly positive. Using the spectral theorem for selfadjoint operators, we conclude that $\square_\varphi^{(0,1)}$ is bijective if $\square_\varphi^{(0,1)}$ is injective (see for instance [W] , (8.17)). In order to show that $\square_\varphi^{(0,1)}$ is injective we consider the inequality

$$(5.3) \qquad\qquad \langle \square_\varphi^{(0,1)} v \, , \, v \rangle \geq \int_{\mathbb{C}^n} \sum_{k=1}^n \lambda_\varphi(z) |v_k(z)|^2 \, d\lambda(z).$$

We recall that $\lambda_\varphi \geq 0$. If $\square_\varphi^{(0,1)} v = 0$, (5.3) together with Assumption (5.1) implies that λ_φ is non zero at ∞, hence $v = 0$ on a non-empty open set. Therefore by the uniqueness result of Kazdan ([Kaz]) it follows that $v = 0$ everywhere and that $\square_\varphi^{(0,1)}$ is injective and therefore also surjective and has a bounded inverse N_φ. Hence we can apply Theorem 3.1. $\qquad\square$

Theorem 5.2.

Let φ be a plurisubharmonic C^2 - function on $\mathbb{C}^n$ such that

$$(5.4) \qquad\qquad \lim_{|z| \to \infty} \lambda_\varphi(z) = +\infty \, .$$

Then the canonical solution operator to $\overline{\partial}$ S^{can} is compact.

Proof.

By Theorem 3.1 and (3.1), it is sufficient to show that S_φ is compact from $L^2_{(0,1)}(\mathbb{C}^n)$ into $L^2(\mathbb{C}^n)$. Using (5.3) and (5.4), it follows that $\square_\varphi^{(0,1)}$ has compact resolvent (see for instance [AHS] or [I]) and we have also shown in Theorem (5.1), that $\square_\varphi^{(0,1)}$ was bijective. The operator N_φ is consequently a compact self-adjoint operator on $L^2_{(0,1)}(\mathbb{C}^n)$.

The operator $S_\varphi = \overline{D}_1^* N_\varphi$ is the canonical solution operator to $\overline{D}_1 v = g$. Now if N_φ is compact, it is standard that $N_\varphi^{\frac{1}{2}}$ is compact. It is then easy to show from (3.5) that S_φ is compact.

$\qquad\square$

Remark 5.3.

Theorem 5.2 can be applied for instance in the case when the weight function is of the form

$$\varphi(z) = \left(\sum_{j=1}^n |z_j|^2 \right)^m \, ,$$

for some integer $m > 1$. This is strongly related to examples given by M. Derridj for the analysis of the regularity of $\square_b$, as discussed in the book [HeNo] (Chap. V.2).

Remark 5.4.

Theorem 5.2 should be compared with the corresponding estimate in [H] (4.4.1.), which is of the form

$$\int_{\mathbb{C}^n} |u(z)|^2 \, e^{-2\varphi(z)} \, d\lambda(z) \leq \int_{\mathbb{C}^n} |\overline{\partial} u(z)|^2 \, \frac{e^{-2\varphi(z)}}{\lambda_\varphi(z)} \, d\lambda(z),$$

for all u in the domain of $\overline{\partial}$ orthogonal to $\ker \overline{\partial}$.

In addition we note that the last inequality is similar to a Brascamp-Lieb inequality as analyzed by Witten-Laplacians techniques (see for example [He3] and the references therein including the generalization obtained by [Jo]).

If 0 is not in the spectrum of $\square_\varphi^{(0,1)}$, then we have

$$(5.5) \qquad \int_{\mathbb{C}^n} |u(z)|^2 \, e^{-2\varphi(z)} \, d\lambda(z) \leq \frac{1}{2} \langle M_\varphi^{-1} \overline{\partial} u \, , \, \overline{\partial} u \rangle_{L^2_{(0,1)}(\mathbb{C}^n,\varphi)},$$

for all u in the domain of $\overline{\partial}$ orthogonal to $\ker \overline{\partial}$.

Let us give the very short proof. By Ruelle's Lemma [Ru], we immediately deduce from (5.2) that

$$N_\varphi \leq \frac{1}{2} M_\varphi^{-1} \, .$$

Now, with $v = u \exp(-\varphi)$ and $g = \overline{D_1} v = \exp(-\varphi) \overline{\partial} u$, we obtain :

$$||v||^2 = \langle v \, , \, S_\varphi g \rangle = \langle g \, , \, N_\varphi g \rangle \leq \frac{1}{2} \langle M_\varphi^{-1} g \, , \, g \rangle \, ,$$

where all the norms and scalar products are in L^2 with the Lebesgue measure. This gives (5.5).

This implies in particular Hörmander's statement above, but not Shigekawa's result below.

Remark 5.5.

In this connection it is also interesting to mention a result of Shigekawa ([Shi]) stating that the space $A^2(\mathbb{C}^n, \varphi)$ is of infinite dimension if the lowest eigenvalue $\lambda_\varphi(z)$ of M_φ satisfies the condition

$$\lim_{|z| \to \infty} |z|^2 \, \lambda_\varphi(z) = \infty \, .$$

This condition implies that 0 is an eigenvalue of infinite multiplicity for a Pauli operator of the form

$$\widetilde{H}(a) = \sum_{j=1}^{2n} (-i\partial_j - a_j(x))^2 + \sum_{j,k=1}^{2n} \frac{i}{2} \, b_{jk}(x) \gamma^j \gamma^k,$$

acting on $L^2(\mathbb{R}^{2n}) \otimes \mathbb{C}^r$, where $b_{jk} = \partial_j a_k - \partial_k a_j$, where $r = 2^n$ and where the γ^j's are the $r \times r$ Dirac matrices satisfying $\gamma^j \gamma^k + \gamma^k \gamma^j = 2\delta^{jk}$, ($\delta^{jk}$ being the Kronecker delta) (see [Shi]).

Shigekawa also analyzes the link between $\widetilde{H}(a)$ and the complex Witten Laplacian by comparing the essential spectra of these operators.

Finally we prove a variant of Theorem 5.2 using the results from [HeMo], together with ideas of M. Derridj (see [HeNo] and references therein).

Theorem 5.6.

If φ is a plurisubharmonic $\mathcal{C}^2$ - function on $\mathbb{C}^n$ and suppose that there exists a number $t \in (0, 1/4)$ and a compact set K in $\mathbb{C}^n$ such that for the Levi matrix M_φ the estimate

$$M_\varphi \geq t\Delta\varphi \otimes I$$

holds outside of K and that λ_φ does not vanish identically. Assume that Δ_φ has compact resolvent. Then the canonical solution operator S operator to $\overline{\partial}$ is compact.

Proof.
Using (3.3), we have :

(5.6) $$\Box_\varphi^{(0,1)} \geq (\Box_\varphi^{(0,0)} + 2t\Delta\varphi) \otimes I \;,$$

outside the compact set K.

By formula (3.6), we are then reduced to the analysis of the compactness of the resolvent of

$$\frac{1}{4}\Delta_\varphi^{(0)} + (2t - 1/4)\Delta\varphi \;.$$

which is reduced, observing that for some constant $C_t > 0$ we have

$$\frac{1}{C_t}\Delta_\varphi^{(0)} \leq \Delta_\varphi^{(0)} + (8t - 1)\Delta\varphi \leq C_t\Delta_\varphi^{(0)} \;,$$

to the same question for $\Delta_\varphi^{(0)}$.

$\square$

We now complete the discussion by saying under which condition $\Delta_\varphi^{(0)}$ has compact resolvent. This has been done already in detail when $n = 1$. One can of course use the criterion of Helffer-Mohamed recalled in the previous section (or some of the improvements obtained later).
Actually, a complementary result can be obtained by generalizing our analysis in $\mathbb{C}$. We observe indeed in the same way as in the case of $\mathbb{C}$, that $\Delta_\varphi^{(0)}$ has compact resolvent if $-\Delta + \Delta\varphi$ has compact resolvent.
This is then the case if we have the conditions that $\Delta\varphi \in A_\infty$ and if

$$\liminf_{|z|\to\infty} \int_{\Pi_{j=1}^n B(z_j,1)} (\Delta\varphi(y))\, d\lambda = +\infty \;.$$

6. THE CASE OF DECOUPLED WEIGHTS

Here we consider weights φ of the form

$$\varphi(z_1,\ldots,z_n) = \sum_{j=1}^n \varphi_j(z_j),$$

where the functions φ_j are C^∞ on $\mathbb{C}$.

6.1. About Dirac and Pauli operators.

In this case an interesting connection to Dirac and Pauli operators is of importance (see [CFKS], [Er], [HNW], [Roz], [Tha]). Let us first consider the real two dimensional case. The Dirac operator $\mathbb{D}$ is defined by

$$\mathbb{D} = \sigma_1 \left(\frac{1}{i}\partial_{x_1} - A_1(x,y)\right) + \sigma_2 \left(\frac{1}{i}\partial_{x_2} - A_2(x,y)\right),$$

where
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

It turns out that the square of $\mathbb{D}$ is diagonal with the Pauli operators $P_\pm$ on the diagonal:
$$\mathbb{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix},$$

where
$$P_\pm = \left(\frac{1}{i} \partial_{x_1} - A_1(x, y) \right)^2 + \left(\frac{1}{i} \partial_{x_2} - A_2(x, y) \right)^2 \pm B(x, y) .$$

Using the computation done in (2.4), we get, having in mind that $\mathcal{S} = \square_\varphi^{(0,0)}$,
$$4 \, \square_\varphi^{(0,0)} = P_- .$$

It is proved in [HNW] (Theorem 1.3) that at least one of the operators $P_\pm$ has non compact resolvent if φ satisfies in $\mathbb{C}$ the following condition (H_r) :
There exists a sequence of disjoint balls B_n of radius ≥ 1 such that (4.4) is satisfied in the union of these balls.

This is in particular the case when the magnetic potentials are polynomials.

Note also the interesting independent result (cf [CFKS]) that the spectra of P_+ and P_- coincide except at 0. So if P_+ has compact resolvent then P_- has its essential spectrum reduced to $\{0\}$.

6.2. **Main results and proofs.**

Our main theorem in this section is the following

Theorem 6.1.
Let $n \geq 2$ and let φ be a decoupled weight such that there exists j such that φ_j satisfies for some $r_j > 0$ the condition (H_{r_j}), then $\square_\varphi^{(0,1)}$ has a non compact resolvent.
Proof.
As observed in [Has3], a simple computation shows that for the decoupled weights
$$\varphi(z_1, \dots, z_n) = \sum_{j=1}^n \varphi_j(z_j)$$

the operator $\square_\varphi^{(0,1)}$ becomes diagonal, each component on the diagonal being
$$(6.1) \qquad\qquad \mathcal{S}_k = \square_\varphi^{(0,0)} + 2 \, \frac{\partial^2 \varphi_k}{\partial z_k \partial \overline{z}_k}.$$

Then the result is based on the following proposition.
Proposition 6.2.
Let $n \geq 2$. Under the assumptions of the theorem on the weight function φ, there always exists a k such that $\mathcal{S}_k$ is not with compact resolvent.

We observe that $\mathcal{S}_k$ can be rewritten in the form
$$4\mathcal{S}_k = \sum_{j \neq k} P_-^{(j)} + P_+^{(k)},$$

where each operator $P_{\pm}^{(\ell)}$ is the previously analyzed Pauli operator in variables the (x_ℓ, y_ℓ). The result is then obtained from the results by Helffer-Nourrigat-Wang recalled in the previous subsection.   □

Remark 6.3.
It is also easy to see that the kernel of $P_{-}^{(\ell)}$ contains all L^2-distributions of the form

$$f(z_\ell) \exp(-\varphi_\ell(z_\ell)),$$

where f is holomorphic and $z_\ell = x_\ell + iy_\ell$.
Hence $\mathcal{S}_k$ has non-compact resolvent, as soon as the space $A^2(\mathbb{C}, \varphi_\ell)$ is of infinite dimension for some $\ell \neq k$. This can be combined with Shigekawa's result, see also the next propositions.

6.3. On a result of G. Schneider.

In the case of decoupled weights, one can extend a remark of G. Schneider ([Sch]) who was considering the case when $\varphi_j(z_j) = |z_j|^{2m}$ for $m > 1$, to show that the canonical solution operator to $\bar{\partial}$ fails to be compact even on the space $A^2_{(0,1)}(\mathbb{C}^n, \varphi)$ of $(0,1)$-forms with holomorphic coefficients.

Proposition 6.4.
Suppose that $n \geq 2$ and that there exists ℓ such that $A^2(\mathbb{C}, \varphi_\ell)$ is infinite dimensional. Suppose also that $1 \in L^2(\mathbb{C}, \varphi_j)$ for all j and that there exists $k \neq \ell$ such that $\frac{\partial^2 \varphi_k}{\partial z_k \partial \bar{z}_k} \in L^2(\mathbb{C}, \varphi_k)$. Then $\mathcal{S}_k$ has non compact resolvent. In particular, $\square_\varphi^{(0,1)}$ has non compact resolvent.
Proof.
Let f_ν an infinite orthonormal system in $A^2(\mathbb{C}, \varphi_\ell)$. For the functions

$$u_\nu(z) = f_\nu(z_\ell) \ \exp(-\varphi(z))$$

we have by (3.2)

$$\square_\varphi^{(0,0)} u_\nu = \overline{D}_1^* \overline{D}_1 u_\nu = 0 \ ,$$

for all $\nu = 1, 2, \ldots$ and by (3.3)

$$\square_\varphi^{(0,1)} (u_\nu \, d\bar{z}_k) = (\mathcal{S}_\ell u_\nu) \, d\bar{z}_k = \left(2 \, \frac{\partial^2 \varphi_k}{\partial z_k \partial \bar{z}_k} u_\nu \right) \, d\bar{z}_k \ .$$

Hence, the sequence

$$\langle \square_\varphi^{(0,1)} (u_\nu \, d\bar{z}_k) \, , \, (u_\nu \, d\bar{z}_k) \rangle = \langle \mathcal{S}_k \, u_\nu \, , \, u_\nu \rangle$$

is bounded and, by the assumption that the functions $z_\ell \mapsto f_\nu(z_\ell) \exp(-\varphi_\ell(z_\ell))$ form an orthonormal system, we get the statement.   □

Using a similar argument we get the following extension of a result of G. Schneider [Sch] (see also [Kr]).
Proposition 6.5.
Suppose that $n \geq 2$ and that there exists ℓ such that $A^2(\mathbb{C}, \varphi_\ell)$ is infinite dimensional. Suppose also that $1 \in L^2(\mathbb{C}, \varphi_j)$ for all j. Suppose finally that for some $k \neq \ell$, $\bar{z}_k \in$

$L^2(\mathbb{C}, \varphi_k)$. *Then the canonical solution operator to $\overline{\partial}$ fails to be compact even on the space $A^2_{(0,1)}(\mathbb{C}^n, \varphi)$.*

Proof.

Let P_k denote the Bergman projection from $L^2(\mathbb{C}, \varphi_k)$ onto $A^2(\mathbb{C}, \varphi_k)$. It is clear that the function $(\overline{z}_k - P_k \overline{z}_k)$ is not zero. With the notations of the preceding proof, the family

$$h_\nu := f_\nu(z_\ell)(\overline{z}_k - P_k \overline{z}_k)$$

is an orthogonal family in $A^2(\mathbb{C}^n, \varphi)^\perp$, which satisfies $\overline{\partial} h_\nu = f_\nu(z_\ell) d\overline{z}_k$.

Hence $(\overline{\partial} h_\nu)_\nu$ constitutes a bounded sequence in $A^2_{(0,1)}(\mathbb{C}^n, \varphi)$, and this implies the result. $\square$

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F. Haslinger: Institut für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria
E-mail address: friedrich.haslinger@univie.ac.at

B. Helffer : Laboratoire de Mathématique, Université Paris Sud, Bâtiment 425, F-91405 Orsay, France
E-mail address: bernard.helffer@math.u-psud.fr